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Staggered eight-vertex model with four sublattices†

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Abstract. We have studied the staggered eight-vertex model on a square lattice which allows different vertex weights for the four sublattices of the square lattice. The soluble case of a free-fermion model is solved by the Pfaffian method. It is shown that our model may exhibit up to five phase transitions. In general the specific heat has logarithmic singularities, except in some special cases where the system exhibits first- or second-order phase transition(s).

1. Introduction

Recently Hsue *et al* (1975) studied the staggered eight-vertex model on a square lattice which allows different vertex weights for the two sublattices. They considered the soluble case of a free-fermion model which can be solved by the Pfaffian method. Their model may exhibit up to three phase transitions. In general the specific heat has logarithmic singularities, except in some special cases where the system exhibits second-order phase transition and the specific heat diverges with an exponent $\alpha = \frac{1}{2}$. The staggered eight-vertex model on a Kagomé lattice‡ was considered by Lin (1976a) and the corresponding free-fermion model may exhibit up to five phase transitions. The Ising model on a Union Jack lattice which has three phase transitions (Vaks *et al* 1965) is equivalent to a special case of the staggered eight-vertex model on a square lattice (Hsue *et al* 1975).

The purpose of this paper is to generalize the results of Hsue *et al* to the staggered eight-vertex model which allows four different vertex weights for the four sublattices of the square lattice. Our model is described in § 2. Symmetry relations are discussed in § 3. When the vertex weights satisfy the free-fermion condition, the model can be solved by the Pfaffian method (Montroll 1964). The Pfaffian solution is given in § 4. There are five cases where the free-fermion condition is satisfied at all temperatures. These cases are discussed in § 5. Our conclusion is given in § 6.

2. Definition of the model

Place arrows on the bonds of a square lattice L of N sites and allow only those configurations with an even number of arrows pointing into each vertex. The four

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‡ This model is equivalent to a special case of the 32-vertex model on a triangular lattice (Lin 1976b).

sublattices of L are denoted by A, B, C, and D, as shown in figure 1. The eight configurations allowed at each vertex are shown in figure 2, where each vertex type is assigned a weight. Let the vertex weights be

$$
\begin{aligned}
\{\omega\} &= \{\omega_1, \omega_2, \ldots, \omega_8\} && \text{on A} \\
\{\omega'\} &= \{\omega'_1, \omega'_2, \ldots, \omega'_8\} && \text{on B} \\
\{\omega''\} &= \{\omega''_1, \omega''_2, \ldots, \omega''_8\} && \text{on C} \\
\{\omega'''\} &= \{\omega'''_1, \omega'''_2, \ldots, \omega'''_8\} && \text{on D.}
\end{aligned} \tag{1}
$$

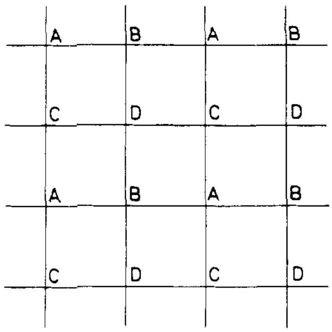

Figure 1. The square lattice with four sublattices A, B, C, and D.

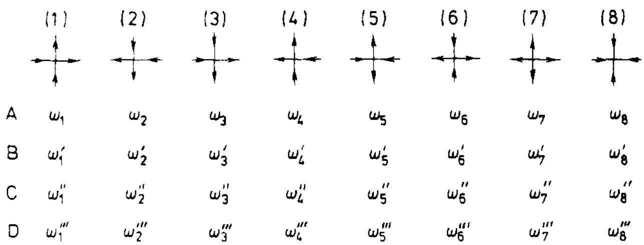

Figure 2. The eight-vertex configurations and the associated vertex weights.

The partition function is

$$
Z = \sum \left(\prod \omega_i^{n_i}\right)\left(\prod \omega_i'^{n'_i}\right)\left(\prod \omega_i''^{n''_i}\right)\left(\prod \omega_i'''^{n'''_i}\right) \tag{2}
$$

where the summation is extended to all allowed arrow configurations on L, and $n_i(n'_i, n''_i, n'''_i)$ is the number of the ith-type sites on A(B, C, D). The goal is to compute the 'free energy'

$$
\psi = \lim_{N\to\infty} \frac{1}{N} \ln Z. \tag{3}
$$

In a physical model, the vertex weights are interpreted as the Boltzmann factors

$$
\omega_i = \exp(-\beta e_i) \qquad \omega'_i = \exp(-\beta e'_i) \qquad \omega''_i = \exp(-\beta e''_i) \qquad \omega'''_i = \exp(-\beta e'''_i) \tag{4}
$$

where $\beta = 1/kT$, k is the Boltzmann constant, T is the temperature, and e_i, e'_i, e''_i, e'''_i are the vertex energies.

3. Symmetry relations

The partition function Z possesses some symmetry relations which follow from general considerations. Translational invariance of the square lattice implies that Z is invariant under each of the following transformations:

$$\begin{array}{lll} (1) & \{\omega\}\leftrightarrow\{\omega'\} & \{\omega''\}\leftrightarrow\{\omega'''\} \\ (2) & \{\omega\}\leftrightarrow\{\omega''\} & \{\omega'\}\leftrightarrow\{\omega'''\} \\ (3) & \{\omega\}\leftrightarrow\{\omega'''\} & \{\omega'\}\leftrightarrow\{\omega''\}. \end{array} \tag{5}$$

We write

$$Z = Z(12345678, 1'2'3'4'5'6'7'8', 1''2''3''4''5''6''7''8'', 1'''2'''3'''4'''5'''6'''7'''8''') \tag{6}$$

where i, i', i'', i''' denote respectively ω_i, ω'_i, ω''_i, ω'''_i.

Reversing all arrows along one direction (horizontal or vertical) we obtain

$$\begin{aligned} Z &= Z(43217856, 4'3'2'1'7'8'5'6', 4''3''2''1''7''8''5''6'', 4'''3'''2'''1'''7'''8'''5'''6''') \\ &= Z(34128765, 3'4'1'2'8'7'6'5, 3''4''1''2''8''7''6''5'', 3'''4'''1'''2'''8'''7'''6'''5'''). \end{aligned} \tag{7}$$

Reversing all arrows implies

$$Z = Z(21436587, 2'1'4'3'6'5'8'7, 2''1''4''3''6''5''8''7'', 2'''1'''4'''3'''6'''5'''8'''7'''). \tag{8}$$

Inversion symmetry implies

$$Z = Z(21435678, 2'1'4'3'5'6'7'8, 2''1''4''3''5''6''7''8'', 2'''1'''4'''3'''5'''6'''7'''8'''). \tag{9}$$

Reflection symmetry implies

$$\begin{aligned} Z &= Z(43215678, 4'3'2'1'5'6'7'8, 4''3''2''1''5''6''7''8'', 4'''3'''2'''1'''5'''6'''7'''8''') \\ &= Z(34125678, 3'4'1'2'5'6'7'8, 3''4''1''2''5''6''7''8'', 3'''4'''1'''2'''5'''6'''7'''8''') \\ &= Z(21346578, 2''1''3''4''6''5''7''8'', 2'1'3'4'6'5'7'8, 2'''1'''3'''4'''6'''5'''7'''8''') \\ &= Z(1'''2'''4'''3'''6'''5'''7'''8''', 1'2'4'3'6'5'7'8, 1''2''4''3''6''5''7''8'', 12436578). \end{aligned} \tag{10}$$

Reversing all arrows along the zigzag paths shown in figure 2 of Fan and Wu (1970) implies

$$Z = Z(65782134, 5'6'8'7'1'2'4'3, 5''6''8''7''1''2''4''3'', 6'''5'''7'''8'''2'''1'''3'''4'''). \tag{11}$$

Finally there is the weak-graph symmetry (Nagle and Temperley 1968) which is a local property of a lattice and is valid even if the weights are site dependent.

4. Pfaffian solution

A vertex model can be solved by the Pfaffian method if the free-fermion condition is satisfied at each vertex (Fan and Wu 1970). In our model, the condition reads

$$\begin{aligned} &\omega_1\omega_2+\omega_3\omega_4=\omega_5\omega_6+\omega_7\omega_8 \\ &\omega'_1\omega'_2+\omega'_3\omega'_4=\omega'_5\omega'_6+\omega'_7\omega'_8 \\ &\omega''_1\omega''_2+\omega''_3\omega''_4=\omega''_5\omega''_6+\omega''_7\omega''_8 \\ &\omega'''_1\omega'''_2+\omega'''_3\omega'''_4=\omega'''_5\omega'''_6+\omega'''_7\omega'''_8. \end{aligned} \tag{12}$$

Under this condition the partition function is equal to a Pfaffian which is evaluated in the appendix. The result is

$$\psi=\frac{1}{16\pi^2}\int_0^{2\pi}\mathrm{d}\theta\int_0^{2\pi}\mathrm{d}\phi\,\ln F(\theta,\phi) \tag{13}$$

where

$$\begin{aligned}F(\theta,\phi)=\Omega_1^2+\Omega_2^2+\Omega_3^2+\Omega_4^2&+2(\Omega_2\Omega_4-\Omega_3\Omega_1)\cos\theta\\&+2(\Omega_2\Omega_3-\Omega_1\Omega_4)\cos\phi+2(\Omega_3\Omega_4-\Omega_5\Omega_6)\cos(\theta+\phi)\\&+2(\Omega_5\Omega_6-\Omega_1\Omega_2)\cos(\theta-\phi)-4a\sin^2\theta-4b\sin^2\phi\\&+4c\sin\theta\sin(\theta+\phi)+4d\sin\phi\sin(\theta+\phi)+4e\sin\theta\sin(\theta-\phi)\\&+4f\sin\phi\sin(\phi-\theta)-4g\sin^2(\theta+\phi)-4h\sin^2(\theta-\phi)\end{aligned}$$

$$\Omega_1=11'1''1'''+22'2''2'''+33'3''3'''+44'4''4'''+68'7''5'''+75'6''8'''+57'8''6'''+86'5''7'''$$

$$\Omega_2=13'4''2'''+24'3''1'''+31'2''4'''+42'1''3'''+56'6''5'''+65'5''6'''+78'8''7'''+87'7''8'''$$

$$\Omega_3=22'3''3'''+33'2''2'''+11'4''4'''+44'1''1'''+57'6''8'''+68'5''7'''+86'7''5'''+75'8''6'''$$

$$\Omega_4=24'2''4'''+42'4''2'''+13'1''3'''+31'3''1'''+56'8''7'''+78'6''5'''+65'7''8'''+87'5''6'''$$

$$\begin{aligned}\Omega_5\Omega_6=(11'1''1'''+22'2''2''')(56'6''5'''+65'5''6''')&+(33'3''3'''+44'4''4''')(78'8''7'''+87'7''8''')\\&+(13'4''2'''+24'3''1''')(57'8''6'''+68'7''5''')\\&+(31'2''4'''+42'1''3''')(75'6''8'''+86'5''7''')\\&+(42'4''2'''+31'3''1''')(57'6''8'''+68'5''7''')\\&+(13'1''3'''+24'2''4''')(75'8''6'''+86'7''5''')\\&+(11'4''4'''+22'3''3''')(78'6''5'''+87'5''6''')\\&+(33'2''2'''+44'1''1''')(56'8''7'''+65'7''8''')\\&+56(1'4'1''3''1'''1'''+2'3'2''4''2'''2''')+12(5'7'5''8''6'''6'''+6'8'6''7''5'''5''')\\&+5'6'(141''1''1'''3'''+232''2''2'''4''')+1'2'(576''6''5'''8'''+685''5''6'''7''')\\&+5''6''(131'1'1'''4'''+242'2'2'''3''')+1''2''(586'6'5'''7'''+675'5'6'''8''')\\&+5'''6'''(111'3'1''4''+222'4'2''3'')+1'''2'''(665'8'5''7''+556'7'6''8'')\\&+34(5'7'6''7''8'''8'''+6'8'5''8''7'''7''')+78(1'4'2''4''4'''4'''+2'3'1''3''3'''3''')\\&+3'4'(578''8''6'''7'''+687''7''5'''8''')+7'8'(144''4''2'''4'''+233''3''1'''3''')\\&+3''4''(678'8'5'''7'''+587'7'6'''8''')+7''8''(244'4'1'''4'''+133'3'2'''3''')\\&+3'''4'''(886'7'5''7''+775'8'6''8'')+7'''8'''(442'4'1''4''+331'3'2''3'')\end{aligned}$$

$$\begin{aligned}a=(12-78)(3'4'-7'8')(1''2''-7''8'')(3'''4'''-7'''8''')&+(34-78)(1'2'-7'8')(3''4''-7''8'')\\&\times(1'''2'''-7'''8''')+(232''3''+141''4''-576''8''-685''7'')\\&\times(2'3'2'''3'''+1'4'1'''4'''-5'7'6'''8'''-6'8'5'''7''')\end{aligned}$$

$$b=(34-78)(3'4'-7'8')(1''2''-7''8'')(1'''2'''-7'''8''')+(12-78)(1'2'-7'8')(3''4''-7''8'')$$
$$\times(3'''4'''-7'''8''')+(242'4'+131'3'-675'8'-586'7')$$
$$\times(2''4''2'''4'''+1''3''1'''3'''-6''7''5'''8'''-5''8''6'''7''')$$

$$c=(12-78)(1''2''-7''8'')(1'4'1'''4'''+2'3'2'''3'''-5'7'6'''8'''-6'8'5'''7''')$$
$$+(1'2'-7'8')(1'''2'''-7'''8''')(141''4''+232''3''-576''8''-685''7'')$$

$$d=(12-78)(1'2'-7'8')(1''3''1'''3'''+2''4''2'''4'''-6''7''5'''8'''-5''8''6'''7''')$$
$$+(1''2''-7''8'')(1'''2'''-7'''8''')(131'3'+242'4'-675'8'-586'7')$$

$$e=(34-78)(3''4''-7''8'')(1'4'1'''4'''+2'3'2'''3'''-5'7'6'''8'''-6'8'5'''7''')$$
$$+(3'4'-7'8')(3'''4'''-7'''8''')(141''4''+232''3''-576''8''-685''7'')$$

$$f=(34-78)(3'4'-7'8')(1''3''1'''3'''+2''4''2'''4'''-6''7''5'''8'''-5''8''6'''7''')$$
$$+(3''4''-7''8'')(3'''4'''-7'''8''')(131'3'+242'4'-675'8'-586'7')$$

$$g=(12-78)(1'2'-7'8')(1''2''-7''8'')(1'''2'''-7'''8''')$$

$$h=(34-78)(3'4'-7'8')(3''4''-7''8'')(3'''4'''-7'''8''')$$

and i, i', i'', i'' denote respectively ω_i, ω_i', ω_i'', ω_i'''.

The special case of $\omega_i=\omega_i'=\omega_i''=\omega_i'''$ has been considered by Fan and Wu (1970). The special case of $\omega_i=\omega_i'''$ and $\omega_i'=\omega_i''$ was considered by Hsue *et al* (1975), in this case it is readily verified that $F(\theta,\phi)$ can be factorized into two factors:

$$F(\theta,\phi)=(A+2B\cos\alpha+2C\cos\beta+2D\cos(\alpha-\beta)+2E\cos(\alpha+\beta)-4\Delta\sin^2\beta$$
$$-4\Delta'\sin^2\alpha][A-2B\cos\alpha-2C\cos\beta+2D\cos(\alpha-\beta)$$
$$+2E\cos(\alpha+\beta)-4\Delta\sin^2\beta-4\Delta'\sin^2\alpha] \qquad (14)$$

where

$$\theta=\alpha-\beta \qquad \phi=\alpha+\beta$$
$$\Delta=(\omega_1\omega_2-\omega_5\omega_6)(\omega_1'\omega_2'-\omega_5'\omega_6') \qquad \Delta'=(\omega_3\omega_4-\omega_5\omega_6)(\omega_3'\omega_4'-\omega_5'\omega_6')$$
$$A=u_2^1+u_2^2+u_3^2+u_4^2 \qquad B=u_1u_3-u_2u_4 \qquad C=u_1u_4-u_2u_3$$
$$D=u_5u_6-u_1u_2 \qquad E=u_3u_4-u_5u_6$$
$$u_1=\omega_1\omega_1'+\omega_2\omega_2' \qquad u_2=\omega_3\omega_3'+\omega_4\omega_4' \qquad u_3=\omega_5\omega_6'+\omega_6\omega_5'$$
$$u_4=\omega_7\omega_8'+\omega_8\omega_7' \qquad u_5u_6=\omega_1\omega_3\omega_1'\omega_3'+\omega_2\omega_4\omega_2'\omega_4'+\omega_5\omega_7\omega_6'\omega_8'+\omega_6\omega_8\omega_5'\omega_7'$$

and our solution (13) indeed reduces to the previously known expressions. The special case of $\omega_7=\omega_8=\omega_7'=\omega_8'=\omega_7''=\omega_8''=\omega_7'''=\omega_8'''=0$ was considered by Lin and Tang (1976).

It is easy to check that $F(\theta,\phi)=0$ at these following points:

$$\begin{aligned}
&\theta=\phi=0 && \Omega_1=\Omega_2+\Omega_3+\Omega_4\\
&\theta=\phi=\pi && \Omega_2=\Omega_1+\Omega_3+\Omega_4\\
&\theta=0,\quad \phi=\pi && \Omega_3=\Omega_1+\Omega_2+\Omega_4\\
&\theta=\pi,\quad \phi=0 && \Omega_4=\Omega_1+\Omega_2+\Omega_3.
\end{aligned} \qquad (15)$$

5. Exactly soluble models

The free-fermion condition (12) can be satisfied at all temperatures provided the vertex energies e_i, e_i', e_i'', e_i''' satisfy some identities. There are five exactly soluble models (others are related to them by symmetry):

$$
\begin{array}{llll}
(1) & e_1+e_2=e_5+e_6 & e_3+e_4=e_7+e_8 & e_1'+e_2'=e_7'+e_8' \\
& e_3'+e_4'=e_5'+e_6' & e_1''+e_2''=e_7''+e_8'' & e_3''+e_4''=e_5''+e_6'' \\
& e_1'''+e_2'''=e_5'''+e_6''' & e_3'''+e_4'''=e_7'''+e_8''' &
\end{array} \qquad (16)
$$

$$
\begin{array}{llll}
(2) & e_1+e_2=e_5+e_6 & e_3+e_4=e_7+e_8 & e_1'+e_2'=e_7'+e_8' \\
& e_3'+e_4'=e_5'+e_6' & e_1''+e_2''=e_5''+e_6'' & e_3''+e_4''=e_7''+e_8'' \\
& e_1'''+e_2'''=e_7'''+e_8''' & e_3'''+e_4'''=e_5'''+e_6''' &
\end{array} \qquad (17)
$$

$$
\begin{array}{llll}
(3) & e_1+e_2=e_5+e_6 & e_3+e_4=e_7+e_8 & e_1'+e_2'=e_5'+e_6' \\
& e_3'+e_4'=e_7'+e_8' & e_1''+e_2''=e_7''+e_8'' & e_3''+e_4''=e_5''+e_6'' \\
& e_1'''+e_2'''=e_7'''+e_8''' & e_3'''+e_4'''=e_5'''+e_6''' &
\end{array} \qquad (18)
$$

$$
\begin{array}{llll}
(4) & e_1+e_2=e_5+e_6 & e_3+e_4=e_7+e_8 & e_1'+e_2'=e_5'+e_6' \\
& e_3'+e_4'=e_7'+e_8' & e_1''+e_2''=e_5''+e_6'' & e_3''+e_4''=e_7''+e_8'' \\
& e_1'''+e_2'''=e_7'''+e_8''' & e_3''+e_4'''=e_5'''+e_6''' &
\end{array} \qquad (19)
$$

$$
\begin{array}{llll}
(5) & e_1+e_2=e_5+e_6 & e_3+e_4=e_7+e_8 & e_1'+e_2'=e_5'+e_6' \\
& e_3'+e_4'=e_7'+e_8' & e_1''+e_2''=e_5''+e_6'' & e_3''+e_4''=e_7''+e_8'' \\
& e_1'''+e_2'''=e_5'''+e_6''' & e_3'''+e_4'''=e_7'''+e_8'''. &
\end{array} \qquad (20)
$$

In general, all the zeros of $F(\theta, \phi)$ are given by (15) and the critical temperature T_c is determined by $\Delta(T_c)=0$ where

$$\Delta(T)=\Omega_1+\Omega_2+\Omega_3+\Omega_3-2\max\{\Omega_1, \Omega_2, \Omega_3, \Omega_4\}. \qquad (21)$$

Since neither ψ nor its derivatives can be expressed in terms of known functions except in special cases, we shall confine our discussion to the critical behaviour of ψ. To be specific, we consider the non-analyticity of ψ at $\Omega_1=\Omega_2+\Omega_3+\Omega_4$. Following Hsue *et al* (1975, equation (49)), we expand $F(\theta, \phi)$ about $\theta=\phi=0$ and get

$$\psi_{\text{singular}} \sim \int_0 \mathrm{d}\theta \int_0 \mathrm{d}\phi \ln[(\Omega_1-\Omega_2-\Omega_3-\Omega_4)^2+p\theta^2+q\theta\phi+r\phi^2] \qquad (22)$$

where

$$p=(\Omega_1-\Omega_4)(\Omega_2+\Omega_3)-4(a-c-e+g+h)$$
$$q=4\Omega_5\Omega_6-2\Omega_1\Omega_2-2\Omega_3\Omega_4+4(c+d-e-f-2g+2h)$$
$$r=(\Omega_1-\Omega_3)(\Omega_2+\Omega_4)-4(b-d-f+g+h)$$

and only the lower integration limits are needed. One finds (Hsue *et al* 1975)

$$\psi_{\text{singular}} \sim (T-T_c)^2 \ln|T-T_c| \qquad \text{if } q^2 \neq 4pr. \qquad (23)$$

The specific heat diverges logarithmically. Since $\Delta(T_c)=0$ may have as much as five solutions, the system may exhibit up to five phase transitions. To see this, let us consider the following case:

$$\begin{aligned}
&e_1=e_2=e_5=e_6=e_4'=e_5'=e_6'=\infty\\
&e_3+e_4=e_7+e_8 \qquad e_1'+e_2'=e_7'+e_8'\\
&e_1''+e_2''=e_7''+e_8'' \qquad e_3''+e_4''=e_5''+e_6''\\
&e_1'''+e_2'''=e_5'''+e_6''' \qquad e_3'''+e_4'''=e_7'''+e_8'''.
\end{aligned}\tag{24}$$

It follows from (24) that we have $a=b=c=d=e=f=g=h=0$ and

$$\begin{aligned}
&\Omega_1=\omega_3\omega_3'\omega_3''\omega_3'''\\
&\Omega_2=\omega_3\omega_1'\omega_2''\omega_4'''+\omega_4\omega_2'\omega_1''\omega_3'''+\omega_7\omega_8'\omega_8''\omega_7'''+\omega_8\omega_7'\omega_7''\omega_8'''\\
&\Omega_3=\omega_3\omega_3'\omega_2''\omega_2'''\\
&\Omega_4=\omega_3\omega_1'\omega_3''\omega_1'''+\omega_4\omega_2'\omega_4''\omega_2'''+\omega_7\omega_8'\omega_6''\omega_5'''+\omega_8\omega_7'\omega_5''\omega_6'''.
\end{aligned}\tag{25}$$

In this case it can be shown (Green and Hurst 1964, § 5.3) that the first derivatives of ψ can be expressed in terms of the complete elliptical integrals of the first and third kinds, consequently the second derivatives of ψ have a logarithmic divergence. The critical condition $\Delta(T_c)=0$ may have five solutions. For example, if

$$\begin{aligned}
&e_3=e_4=e_7=e_8=e_1'=e_2'=e_3'=e_7'=e_8'=e_3''=e_4''=e_5''=e_6''=e_3'''=e_8'''=0\\
&e_1''=e_7''=0{\cdot}15\epsilon \qquad e_2''=e_8''=19\epsilon \qquad e_1'''=e_2'''=e_4'''=e_5'''=e_6'''=e_7'''=\epsilon,
\end{aligned}\tag{26}$$

then we have

$$\begin{aligned}
&\Omega_1=1\\
&\Omega_2=2(e^{-0{\cdot}15\beta\epsilon}+e^{-20\beta\epsilon})\\
&\Omega_3=e^{-20\beta\epsilon}\\
&\Omega_4=4e^{-\beta\epsilon}
\end{aligned}\tag{27}$$

and the critical condition implies

$$\begin{aligned}
&\Omega_1=\Omega_2+\Omega_3+\Omega_4 \qquad \text{at } \beta\epsilon=4{\cdot}8\\
&\Omega_2=\Omega_1+\Omega_3+\Omega_4 \qquad \text{at } \beta\epsilon=4{\cdot}3,\ 2{\cdot}2\\
&\Omega_4=\Omega_1+\Omega_2+\Omega_3 \qquad \text{at } \beta\epsilon=0{\cdot}32,\ 0{\cdot}07.
\end{aligned}\tag{28}$$

The condition $q^2=4pr$ implies that there exist zeros of $F(\theta,\phi)=0$ which are not given by (15) (Hsue *et al* 1975). In this case the system exhibits first- or second-order phase transition. To see this, let us consider the following two cases.

$$\begin{aligned}
(1)\quad &e_1=e_2=e_5=e_6=e_3'=e_4'=e_5'=e_6'=e_3''=e_6''=\infty\\
&e_3=e_4=e_7=e_8=e_1'=e_2'=e_7'=e_8'=e_1''=e_2''=e_4''=e_5''=e_7''=e_8''=e_3'''=0\\
&e_1'''=e_2'''=e_5'''=e_6'''=\epsilon \qquad e_7'''=e_8'''=\tfrac{1}{2}e_4'''=5\epsilon
\end{aligned}\tag{29}$$

$$(2)\quad e_7=e_8=e_7'=e_8'=e_7''=e_8''=e_7'''=e_8'''=\infty.\tag{30}$$

In the first case we have

$$F(\theta,\phi)=\Omega_2^2+\Omega_4^2+2\Omega_2\Omega_4\cos\theta\tag{31}$$

where

$$\Omega_2 = (1+e^{-5\beta\epsilon})^2 \qquad \Omega_4 = 2\, e^{-\beta\epsilon}.$$

It follows from equation (31) that

$$\psi = \tfrac{1}{4} \ln \max\{\Omega_2, \Omega_4\} \tag{32}$$

where the first derivative of ψ has a jump discontinuity at $\Omega_2 = \Omega_4$ (first-order phase transition) and the critical condition $\Omega_2 = \Omega_4$ has two solutions at $\beta\epsilon = 0{\cdot}31$ and $0{\cdot}59$. The second case is the staggered ice-rule vertex model which has been examined in detail by Lin and Tang (1976). In the second case the system may exhibit up to three second-order phase transitions and the specific heat diverges with an exponent $\frac{1}{2}$ either above or below each transition temperature.

6. Conclusion

We have generalized the results of Hsue *et al* (1975) to the staggered eight-vertex model on a square lattice which allows four different vertex weights for the four sublattices. There are five different cases where the vertex weights satisfy the free-fermion condition at all temperatures. We have considered these soluble models and found that the system may exhibit up to five phase transitions. In general the specific heat has logarithmic singularities except in some special cases where the system may exhibit up to two first-order phase transitions or up to three second-order phase transitions (α or $\alpha' = \frac{1}{2}$ at each transition temperature)†.

Appendix. Pfaffian solution

Expand each site of L into a 'city' of four terminals to form a dimer lattice L^{Δ} whose unit cell is shown in figure 3. Following the same procedure as Hsue *et al* (1975), we have

$$\psi = \frac{1}{16\pi^2} \int_0^{2\pi} d\theta \int_0^{2\pi} d\phi \, \ln[(\omega_2 \omega_2' \omega_2'' \omega_2''')^2 D(\theta, \phi)] \tag{A.1}$$

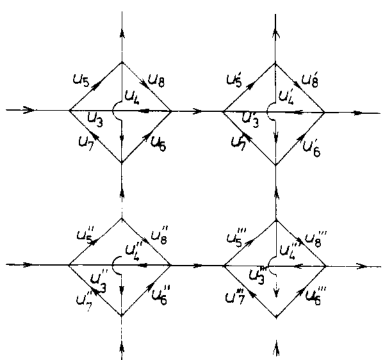

Figure 3. A unit cell of the dimer lattice L^{Δ}.

† We use the standard definitions of critical-point exponents (Stanley 1971).

where

$$D(\theta,\phi)=\left|\begin{array}{cccc|cccc|cccc|cccc}
0 & u_3 & -u_8 & -u_5 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
-u_3 & 0 & u_6 & -u_7 & -e^{-i\theta} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
u_8 & -u_6 & 0 & u_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & e^{i\phi} & 0 & 0 & 0 & 0\\
u_5 & u_7 & -u_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0\\
\hline
0 & e^{i\theta} & 0 & 0 & 0 & u'_3 & -u'_8 & -u'_5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
-1 & 0 & 0 & 0 & -u'_3 & 0 & u'_6 & -u'_7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & u'_8 & -u'_6 & 0 & u'_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & e^{i\phi}\\
0 & 0 & 0 & 0 & u'_5 & u'_7 & -u'_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0\\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & u''_3 & -u''_8 & -u''_5 & 0 & 1 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -u''_3 & 0 & u''_6 & -u''_7 & -e^{-i\theta} & 0 & 0 & 0\\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & u''_8 & -u''_6 & 0 & u''_4 & 0 & 0 & 0 & 0\\
0 & 0 & -e^{-i\phi} & 0 & 0 & 0 & 0 & 0 & u''_5 & u''_7 & -u''_4 & 0 & 0 & 0 & 0 & 0\\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & e^{i\theta} & 0 & 0 & 0 & u'''_3 & -u'''_8 & -u'''_5\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -u'''_3 & 0 & u'''_6 & -u'''_7\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & u'''_8 & -u'''_6 & 0 & u'''_4\\
0 & 0 & 0 & 0 & 0 & 0 & -e^{-i\phi} & 0 & 0 & 0 & 0 & 0 & u'''_5 & u'''_7 & -u'''_4 & 0
\end{array}\right|$$

and

$$u_i=\omega_i/\omega_2 \qquad u'_i=\omega'_i/\omega'_2 \qquad u''_i=\omega''_i/\omega''_2 \qquad u'''_i=\omega'''_i/\omega'''_2 .$$

Equation (A.1) reduces to equation (13) in the text after some algebra.

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